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On the determination of the zones of influence and dependence for three-dimensional boundary-layer equations

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The zones of influence and dependence for three-dimensional boundary-layer equations first studied by Raetz are re-examined from the viewpoint of the subcharacteristics. It is shown that in contrast, the zones of influence and dependence for a totally hyperbolic system are determined by the characteristics; for the present parabolic system of three-dimensional boundary-layer equations, the zones are determined by the characteristics and subcharacteristics. The same idea should be applicable to more general systems of equations of similar type.

1. Introduction

In the classical theory of partial differential equations (see, for example, Courant 1962), the zones of influence and dependence for general second-order partial differential equations with two independent variables are well known. Apart from some special cases, extension of the same idea to equations of higher order and more than two independent variables has been limited to two limiting types of problems, the totally hyperbolic and totally elliptic. These two limiting cases are defined by the roots of the characteristic equation being either all real and distinct or being all imaginary. As for the more general intermediate cases, for example, the case of multiple zero roots, there is no known method for complete determination of the zones of influence and dependence from classical theory of partial differential equations, at least, to the author's knowledge. Among this latter category of fluid mechanics interest is the system of three-dimensional boundary-layer equations.

The characteristics of three-dimensional boundary-layer equations were first investigated by Raetz (1957) and later by Der & Raetz (1962). Their physical picture of the zones of influence and dependence is easy to understand, but Raetz's (1957) mathematical derivations were not so clear. The purpose of this work is to present a different derivation based on the concept of subcharacteristics. The present derivation is believed to be simpler and more instructive. The idea of the zones of influence and dependence is of prime importance to the computation of three-dimensional boundary layers.

The concept of subcharacteristics was introduced in connexion with the singular perturbation problem (Lagerstrom, Cole & Trilling 1949; Cole 1968). Our motivation in relating the boundary-layer problem to the singular perturbation problem should not cause surprise because this is where the subject of singular perturbation was first started. However we shall show that the subcharacteristics studied by these authors determine only the substructure of the flow and hence play a secondary role; while the subcharacteristics considered here share a major role with the characteristics in determining the main structure of the flow. In contrast, the zones of influence and dependence for a totally hyperbolic system are determined by the characteristics; for the present parabolic system, the zones are seen to be determined by the characteristics and subcharacteristics.

2. Equations

Although our purpose is to study the characteristics and the related zones of influence and dependence of the boundary-layer equations, we choose to begin by determining those of the complete Navier–Stokes equations and deduce from them those for the boundary-layer case. This approach provides instructive comparisons and helps to understand the subject better.

For simplicity, we consider the incompressible Navier–Stokes system of equations in Cartesian co-ordinates.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0, \tag{1a}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} = -\frac{\partial p}{\rho\partial x}+\nu\left(\left[\frac{\partial^2 u}{\partial x^2}\right]+\left[\frac{\partial^2 u}{\partial y^2}\right]+\frac{\partial^2 u}{\partial z^2}\right), \tag{1b}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} = -\frac{\partial p}{\rho\partial y}+\nu\left(\left[\frac{\partial^2 v}{\partial x^2}\right]+\left[\frac{\partial^2 v}{\partial y^2}\right]+\frac{\partial^2 v}{\partial z^2}\right), \tag{1c}$$

$$\left[u\frac{\partial w}{\partial x}\right]+\left[v\frac{\partial w}{\partial y}\right]+\left[w\frac{\partial w}{\partial z}\right] = -\frac{\partial p}{\rho\partial z}+\nu\left(\left[\frac{\partial^2 w}{\partial x^2}\right]+\left[\frac{\partial^2 w}{\partial y^2}\right]+\left[\frac{\partial^2 w}{\partial z^2}\right]\right), \tag{1d}$$

where the square brackets indicate the terms which disappear in the boundary-layer approximation. x and y are parallel and z normal to the body surface, u, v and w are the corresponding velocities, p pressure, ρ density and ν kinematic viscosity.

3. Characteristics

To study the characteristics, only the highest derivative terms in each equation of the system are of concern. If Ω denotes the characteristic surface, then the corresponding characteristic determinant, Q, for the systems ($1\,a$–d) can be shown to be (see Petrovsky 1954, p. 33; or Courant 1962, p. 173)

$$Q = \begin{vmatrix} \frac{\partial\Omega}{\partial x} & \frac{\partial\Omega}{\partial y} & \frac{\partial\Omega}{\partial z} & 0 \\ \Delta & 0 & 0 & \left[\frac{\partial\Omega}{\rho\partial x}\right] \\ 0 & \Delta & 0 & \left[\frac{\partial\Omega}{\rho\partial y}\right] \\ 0 & 0 & [\Delta] & \left[\frac{\partial\Omega}{\rho\partial z}\right] \end{vmatrix} \tag{2a}$$

where $$\Delta = -\nu\left\{\left[\left(\frac{\partial\Omega}{\partial x}\right)^2\right]+\left[\left(\frac{\partial\Omega}{\partial y}\right)^2\right]+\left(\frac{\partial\Omega}{\partial z}\right)^2\right\}$$

and the square brackets again indicate the terms which disappear in the boundary-layer approximation. The terms on the first row originate from the continuity equation, and the terms along the last column come from the contribution of the pressure gradients. Δ is obviously associated with the diffusion terms. Upon expansion, (2*a*) yields

$$Q = \begin{cases} \underset{\text{elliptic}}{\overset{\text{incompressibility}}{\dfrac{1}{\rho}\left\{\left(\dfrac{\partial\Omega}{\partial x}\right)^2+\left(\dfrac{\partial\Omega}{\partial y}\right)^2+\left(\dfrac{\partial\Omega}{\partial z}\right)^2\right\}}}\;\underset{\text{elliptic}}{\overset{\text{diffusion}}{\Delta^2}} & \text{Navier–Stokes} \quad (2b) \\ \underset{\text{parabolic}}{\overset{\text{continuity}}{\left(\dfrac{\partial\Omega}{\partial z}\right)}}\;\underset{\text{parabolic}}{\overset{\text{diffusion}}{\left\{\nu\left(\dfrac{\partial\Omega}{\partial z}\right)^2\right\}^2}} & \text{Boundary layer.} \quad (2c) \end{cases}$$

Equations (2*b*, *c*) express the known property that the Navier–Stokes equations are elliptic, while the boundary-layer equations are parabolic. This is because the characteristic equation (setting $Q = 0$) has no real root in the case of the Navier–Stokes equations, but all five roots are real and equal in the boundary-layer case.

$\partial\Omega/\partial z = 0$ implies that all surfaces $\Omega(x,y) = 0$ normal to the body surface are characteristic surfaces. It also indicates that the speed of disturbances is infinite in the z direction. However, nothing is said about the characteristics in the x and y directions. Hence, on the basis of the characteristics alone, propagation in the x and y directions is unlimited. It is at this juncture that classical theory of characteristics does not tell us how to proceed further. In the next section, we propose to resolve this question from the viewpoint of subcharacteristics.

Before we turn to the subcharacteristics, we would like to point out certain features of (2*b*, *c*). Equation (2*b*) has two factors, each of which is of the same elliptic nature, but each has a different physical meaning. The first factor is caused by a combination of the continuity and pressure terms and therefore expresses the property of 'incompressibility' associated with an incompressible Navier–Stokes system. The second factor of (2*b*) represents the diffusion effect.

During the change-over to the boundary-layer case, the difference between the second factors of (2*b*) and (2*c*) is obvious, namely diffusion along directions parallel to the surface is neglected; but the difference between the first factors of (2*b*) and (2*c*) brings out a concept which has thus far not been widely appreciated. The first factor of (2*c*) arises from the continuity equation, but has nothing to do with the pressure. Therefore one cannot relate it in any way to the idea of incompressibility, since the latter is defined by the ratio of the changes in pressure and density. Although it is well known that pressure is assumed given in the boundary-layer theory, what is less well known is that the boundary-layer approximation makes an 'incompressible' boundary-layer flow no longer have the property of 'incompressibility'.

Reduction from (2*b*) to (2*c*) thus reveals that both physical processes – lateral diffusion and incompressibility (or infinite sound speed) – which may generate the upstream influence are lost in the boundary layer. This is a more physical argument than the bare observation that the governing equations become parabolic.

4. Subcharacteristics

The subcharacteristics of the system are obtained by neglecting the viscous terms in ($1b, c, d$). The subcharacteristics so obtained are just the characteristics of the corresponding inviscid equations. For the Navier–Stokes system, the inviscid counterpart is the Euler equations. The required characteristic determinant is

$$Q = \begin{vmatrix} \frac{\partial\Omega}{\partial x} & \frac{\partial\Omega}{\partial y} & \frac{\partial\Omega}{\partial z} & 0 \\ \Delta^* & 0 & 0 & \left[\frac{\partial\Omega}{\rho\partial x}\right] \\ 0 & \Delta^* & 0 & \left[\frac{\partial\Omega}{\rho\partial y}\right] \\ 0 & 0 & [\Delta^*] & \left[\frac{\partial\Omega}{\rho\partial z}\right] \end{vmatrix} \qquad (3a)$$

where $$\Delta^* = \left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)\Omega$$

and the square brackets as before indicate the boundary-layer approximation. Δ^* is now associated with the convection terms. From ($3a$)

$$Q = \begin{cases} \underbrace{\overbrace{\frac{1}{\rho}\left\{\left(\frac{\partial\Omega}{\partial x}\right)^2+\left(\frac{\partial\Omega}{\partial y}\right)^2+\left(\frac{\partial\Omega}{\partial z}\right)^2\right\}}^{\text{incompressibility}}}_{\text{elliptic}} \quad \overset{\text{convection}}{\underset{\text{hyperbolic or parabolic}}{(\Delta^*)^2}} & \text{Euler} \qquad (3b) \\ \overset{\text{continuity}}{\underset{\text{parabolic}}{\left(\frac{\partial\Omega}{dz}\right)}} \quad \overset{\text{convection}}{\underset{\text{hyperbolic or parabolic}}{(\Delta^*)^2}} & \text{Boundary layer.} \qquad (3c) \end{cases}$$

The first factor of ($3b$) is the same as that of ($2b$), and hence represents the elliptic nature. The second factor of ($3b$) indicates that the streamlines are subcharacteristics. They are neither hyperbolic nor parabolic in the usual sense because the two real roots are neither different nor equal to zero. Similar remarks apply to the corresponding boundary-layer case of ($3c$). In particular, we note that although the streamlines are the subcharacteristics for both the Navier–Stokes equations and the boundary-layer equations, they play a different role in the two cases.

When the characteristic equation such as ($3b$) contains factors of different nature, elliptic and others, the elliptic one is dominant. The reason for this will become apparent in the next section. Euler equations, therefore, are elliptic.

5. Zones of influence and dependence

The streamlines as subcharacteristics carry a disturbance with the flow, that is to say, a disturbance is merely convected with finite local velocity along the streamlines. In the case of Navier–Stokes equations, a disturbance is also transferred instantly in all directions through diffusion or because of incompressibility.

This is because both the signal velocity of diffusion and the sound velocity of an incompressible medium are infinite. Due to the vast difference in speed, a disturbance transferred through diffusion or the infinite sound velocity always overtakes that moved by convection. As a consequence, a disturbance at any point affects the entire flow field. The subcharacteristics or streamlines are present, but they do not play a role in determining the main flow structure.

In contrast, in the case of the boundary-layer equations, the subcharacteristics or streamlines do play a role in determining the main flow structure. Here diffusion prevails in the normal (to the body surface) direction only, while the incompressibility property also gets lost in the process of the boundary-layer approximation. A disturbance carried by a streamline will hence not be overtaken along the two surface-parallel directions. Consequently a disturbance at a point P (figure 1) first affects instantly the normal line AB through P, and then is

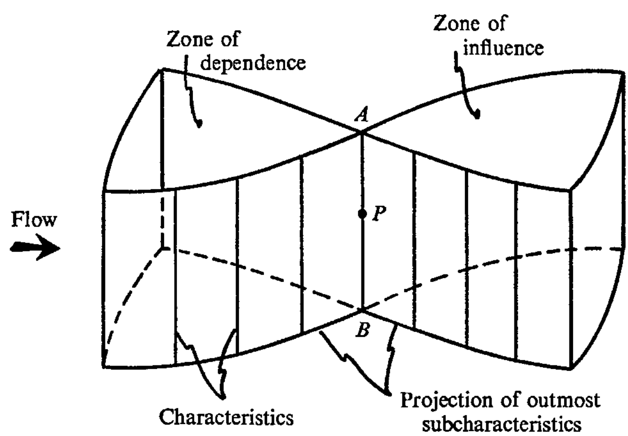

FIGURE 1. Zones of influence and dependence.

convected downstream by all streamlines crossing AB. The zone of influence is a wedge-shaped region bounded by two characteristic surfaces each containing an outmost subcharacteristic or streamline crossing AB. Similarly the zone of dependence for point P is the corresponding wedge-shaped region facing upstream. Thus we arrive at the same picture as Raetz but from a different viewpoint.

The preceding discussion applies also to two-dimensional boundary layers simply by neglecting the derivatives $\partial/\partial y$. Now all the streamlines are confined to the (x, z) plane and, as the wedge angle of figure 1 shrinks to zero, the zones of influence and dependence of a point P become the entire regions downstream and upstream of APB.

Raetz considered that the three-dimensional boundary-layer equations are an elliptic system in the z-direction and a hyperbolic system in the other directions, while in all two-dimensional cases, these equations degenerate to a parabolic system. Although the label-designation is not important, to so characterize the difference between two- and three-dimensional cases may, nevertheless, readily be misunderstood. The implications seem to be that the three-dimensional boundary-layer equations are not 'parabolic', while the two-dimensional equations do not have the property of being elliptic in one direction and hyperbolic in another. Actually this is not the case. Even in the two-dimensional case, the system of equations still may be looked on as elliptic in one direction and hyperbolic in the other. Such an interpretation for the classical heat equation is discussed by Sommerfeld (1949).

6. Practical application

The practical applications of the zones of influence and dependence in the usual supersonic aerodynamics can be used analogously to the three-dimensional boundary layer. In the following, we would like to mention two examples.

One is concerned with the finite difference solutions for three-dimensional boundary layers. In obtaining these numerical solutions, the concept of the zones of influence and dependence is essential. The rationale for this is embraced in the so-called Courant–Friedrichs–Lewy condition (see, for example, Issacson & Keller 1966) for the convergence of the solution of the difference equations to that of the corresponding differential equation.

Unlike the usual linearized supersonic flow where these zones of influence and dependence can be determined beforehand, in the non-linear boundary-layer flow these zones vary from point to point and are not known until the solution is obtained. Rigorous implementation of the rule of these zones in the boundary-layer case greatly complicates the computational procedure. To a large extent, consideration of this rule may decide the method of computation (Hall 1967, Wang 1969) one should use.

Ignoring the zone of dependence in actual calculation may lead to serious questions of convergence and/or stability. Dwyer (1970) in his attempt to calculate the boundary layer over a spinning cone encountered numerical instability when the zone of dependence was first ignored. For a certain class of linear problem, it has been shown (see Isaacson & Kellar) that stability and convergence imply each other. For non-linear problems such as the boundary-layer case, little is known on such equivalence theory. It is not clear whether one can take the convergence for granted, even if no numerical instability ever occurred. The works of Der (1969) and Dwyer & McCroskey (1970) are just such examples. Lack of a complete theory at the present time makes it difficult to say just what effects, if any, the violation of the rule may have on those calculated solutions. In any case, it will be surely better if the rule is followed throughout the computation.

The concept of the zones of dependence and influence is useful also to the understanding of intricate questions. As an example, consider the symmetry-plane boundary-layer problem recently considered by Wang (1970*a*). This type of problem was initiated by Moore (1953) for a supersonic cone. However, the validity of this approach has often been questioned; i.e. can one really solve the symmetry-plane boundary layer independent of the adjacent area? A clear-cut explanation of this plausible question is otherwise difficult to give, but an easy answer can be found by simply pointing out that the zone of dependence is satisfied there (Wang 1970*b*). For the symmetry plane, the general wedge-shaped zone of dependence for a three-dimensional boundary layer degenerates into the symmetry plane itself.

7. New role of subcharacteristics

The problems considered previously by Lagerstrom *et al.* (1949) and Cole (1968) to illustrate the theory of subcharacteristics are of the form

$$\epsilon\phi_{xxt} = \phi_{tt} - \phi_{xx}, \tag{4a}$$

$$a\phi_x + b\phi_t = \epsilon(\phi_{tt} - \phi_{xx}), \tag{4b}$$

where the subscripts denote differentiation, ϵ is a small perturbation parameter, and a, b are constants. Equation (4*a*) has the lines t = constant as characteristics and the lines $x \pm t$ = constant as the subcharacteristics; (4*b*) has the lines $x \pm t$ = constant as characteristics and the lines $bx - at$ = constant as the subcharacteristics. The zones of dependence and influence are determined by the characteristics. The subcharacteristics determine only the substructure of the flow; they gradually gain prominence only when the perturbation parameter ϵ approaches zero.

In the present boundary-layer problem, the situation is different. The zones of dependence and influence are not determined by the characteristics alone. The subcharacteristics are not just playing a secondary role, instead they share the major role with the characteristics in completing the determination of the main flow structure. In this respect, the present study exhibits a new aspect of the theory of subcharacteristics.

8. Possible extensions

Firstly we remark that equations (2)–(3) for the characteristics and subcharacteristics hold also for (1 *a–d*) in general curvilinear co-ordinate systems. One needs only to replace the derivative terms like $\partial/\partial x$ by $\partial/h_1\partial x_1$ where h_1 is the metric coefficient along the x_1 direction, similarly for the other two directions.

Secondly, although our present discussion of using the subcharacteristics to complete the determination of the zones of influence and dependence is confined to the boundary-layer equations, the same idea can be obviously extended to more general partial differential equations.

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